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Weak Bisimulation for Action-Type Coalgebras (Extended Abstract)

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Abstract

We propose a coalgebraic definition of weak bisimulation for a class of coalgebras obtained from bifunctors over the category Set. Weak bisimilarity for a system is obtained as strong bisimilarity of a transformed system. The transformation consists of two steps: First, the behaviour on actions is expanded to behaviour on finite words. Second, the behaviour on finite words is taken modulo the hiding of invisible actions, yielding behaviour on equivalence classes of words closed under silent steps. The coalgebraic definition is justified by two correspondence results, one for the classical notion of weak bisimulation of Milner and another for the notion of weak bisimulation for generative probabilistic transition systems as advocated by Baier and Hermanns.

Keywords: coalgebra, bisimulation, weak bisimulation, labeled transition system, generative probabilistic transition system

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1 Introduction

In this paper we present a definition of weak bisimulation for action-type systems. A typical example of an action-type system is the familiar labelled transition system (LTS) (see, e.g., [20,18]), but also many types of probabilistic systems (see, e.g., [16,27,11,3,26]) fall into this class. In order to emphasize the role of the actions we view coalgebras as arising from bifunctors over the category Set.

In the verification of properties of a system, strong bisimilarity is often too strong an equivalence. Weak bisimilarity [17,18] is a looser equivalence on systems that abstracts away from invisible steps. It is well-known that, in the concrete case of weak bisimilarity for a labelled transition system $\mathcal{S}$, amounts to strong bisimilarity on the 'double-arrowed' system $\mathcal{S}''$ induced by $\mathcal{S}$. We exploit this idea in formulating a general coalgebraic definition of weak bisimilation. Our approach, given a system $\mathcal{S}$, consists of two stages:

(i) First, we define a '$*$-extension' $\mathcal{S}'$ of $\mathcal{S}$ which is a system with the same state set as $\mathcal{S}$, but with action set A^*, the set of all words over A. The system $\mathcal{S}'$ captures the behaviour of $\mathcal{S}$ on finite traces.

(ii) Next, we fix a set of invisible actions $\tau \subseteq A$ and transform $\mathcal{S}'$ into a 'weak-τ-extension' $\mathcal{S}''$ which abstracts away from τ steps. Then we define weak bisimilarity on $\mathcal{S}$ as strong bisimilarity on the weak-τ-extension $\mathcal{S}''$.

In the context of concrete probabilistic transition systems, there have been several proposals for a notion of weak bisimulation, often relying on the particular model under consideration. Segala [27,26] proposed four notions of weak relations for his model of simple probabilistic automata. Baier and Hermanns [3,2,4] have given a rather appealing definition of weak bisimulation for the case of generative probabilistic systems. Philippou, Lee and Sokolsky [21] studied weak bisimulation in the setting of the alternating model [14]. This work was extended to infinite systems by Desharnais, Gupta, Jagadeesan and Panangaden [9]. Desharnais et al. also provided a metric analogue of weak bisimulation [8].

Here, we work in a coalgebraic framework and use the general coalgebraic apparatus of bisimulation [1,15,25]. For weak bisimulation in this setting, there has been early work by Rutten on weak bisimulation for while programs [24] succeeded by a syntactic approach to weak bisimulation by Rothe [23]. In the latter paper, weak bisimulation for a particular class of

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coalgebras was obtained by transforming a coalgebra into an LTS and making use of Milner's weak bisimulation there. This approach also allowed for a definition of weak homomorphisms and weak simulation relations. Later, in the work of Rothe and Mašulović [22] a complex, but interesting coalgebraic theory was developed leading to a notion of weak bisimulation for functors that weakly preserve pullbacks. They also consider a chosen 'observer' and hidden parts of a functor. However, in the case of probabilistic and similar systems, the definition of weak bisimulation does not lead to intuitive results and can not be related to the concrete notions of weak bisimulation mentioned above. The so-called skip relations used in [22] seem to be the major obstacle, as it remains unclear how quantitative information can be incorporated their framework.

The two-phase approach of defining weak bisimilarity advocated in the present paper, amplifying Milner's original idea, comes rather natural. In the category theoretical setting it has been suggested in the context of an open map treatment of weak bisimulation on presheaf models in [10]. However, the approach taken in this paper yields a rather basic and intuitive notion of weak bisimulation. Moreover, not only for the case of labelled transition systems, but also for probabilistic systems the present coalgebraic proposal corresponds to the concrete definitions. Despite the appeal of the coalgebraic definition for weak bisimulation, proofs of correspondence results may vary from straightforward to technically involved. For example, the relevant theorem for labelled transition systems takes less than a page in full in the technical report [29], whereas proving the correspondence result for generative probabilistic systems takes around 20 pages (additional machinery included).

The paper is organized as follows: In Section 2 we provide the basic definitions and properties of the systems under consideration. Section 3 presents the coalgebraic definition of weak bisimulation. We show that our notion of weak bisimilarity leads to Milner's weak bisimilarity for LTSs in Section 4. Section 5 is devoted to a correspondence result for the class of generative systems with respect to the definition of weak bisimilarity of Baier and Hermanns, on the one hand, and our coalgebraic definition, on the other hand. Finally, Section 6 wraps up with some concluding remarks.

2 Systems and bisimilarity

From a coalgebraic point of view [15,12,25], a system is a coalgebra of a given endofunctor, often on the category Set. However, for our approach to defining weak bisimilarity, it is essential to explicitly specify the set of executable actions. Therefore, we shall start from a bifunctor instead of an endofunctor on Set, cf. [5].

A bifunctor is any functor $\mathcal{F}\colon \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set}$. If $\mathcal{F}$ is a bifunctor and A is a fixed set, then the Set endofunctor $\mathcal{F}_A$ is defined by

$$\mathcal{F}_A S = \mathcal{F}(A, S) \quad \text{and} \quad \mathcal{F}_A f = \mathcal{F}\langle id_A, f\rangle, \tag{1}$$

for a set S and a mapping $f\colon S \to T$.
For further reference we state the following simple property.

Proposition 2.1 *Let $\mathcal{F}$ be a bifunctor, A_1, A_2 two sets and $f\colon A_1 \to A_2$ a mapping. Then f induces a natural transformation $\eta^f : \mathcal{F}_{A_1} \Rightarrow \mathcal{F}_{A_2}$ given by $\eta^f_S = \mathcal{F}\langle f, id_S\rangle$.* □

For a bifunctor $\mathcal{F}$, sets S and A and a mapping $\alpha\colon S \to \mathcal{F}_A(S)$, the triple $\langle S, A, \alpha\rangle$ is called an $\mathcal{F}_A$ coalgebra. A homomorphism between two $\mathcal{F}_A$ coalgebras $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ is a function $h\colon S \to T$ satisfying $\mathcal{F}_A h \circ \alpha = \beta \circ h$. The $\mathcal{F}_A$ coalgebras together with their homomorphisms form a category, which we denote by $\mathsf{Coalg}^A_{\mathcal{F}}$.

Definition 2.2 Let $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ be two $\mathcal{F}_A$ coalgebras. A bisimulation between $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ is a relation $R \subseteq S \times T$, such that there exists a mapping $\gamma\colon R \to \mathcal{F}_A R$ making the projections π_1 and π_2 coalgebra homomorphisms between the respective coalgebras,

$$\begin{tikzcd}
S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "\exists\gamma"'] & T \arrow[d, "\beta"] \\
\mathcal{F}_A S & \mathcal{F}_A R \arrow[l, "\mathcal{F}_A\pi_1"] \arrow[r, "\mathcal{F}_A\pi_2"'] & \mathcal{F}_A T
\end{tikzcd}$$

i.e. making the two squares in the above diagram commute. Two states $s \in S$ and $t \in T$ are bisimilar, notation $s \sim t$, if they are related by some bisimulation between $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$.

Let $\mathcal{F}_A$ and $\mathcal{G}_A$ be two Set functors, and let $\eta\colon \mathcal{F}_A \Rightarrow \mathcal{G}_A$ be a natural transformation. The natural transformation η induces a functor $\mathcal{T}\colon \mathsf{Coalg}^A_{\mathcal{F}} \to \mathsf{Coalg}^A_{\mathcal{G}}$ given by

$$\mathcal{T}(\langle S, A, \alpha\rangle) = \langle S, A, \eta_S \circ \alpha\rangle \text{ and } \mathcal{T}(f) = f. \tag{2}$$

Functors induced by natural transformations preserve homomorphisms (cf. [25]) and thus preserve bisimulation relations and bisimilarity.

Next, we present two basic types of systems, labelled transition systems and generative systems, which will be the leading examples in the sequel. We give their concrete definitions first, as well as their corresponding concrete definitions of bisimulation relations, cf. [17,18,16,11].

A labelled transition system, or LTS for short, is a triple $\langle S, A, \rightarrow \rangle$ where S is the set of states, A is the set of actions and $\rightarrow\ \subseteq S \times A \times S$ is the transition relation. As usual, we denote $s \xrightarrow{a} s'$ whenever $\langle s, a, s' \rangle \in\ \rightarrow$.

Definition 2.3 Let $\langle S, A, \rightarrow \rangle$ be an LTS. An equivalence relation $R \subseteq S \times S$ is a strong bisimulation on $\langle S, A, \rightarrow \rangle$ if and only if, for every pair $\langle s, t \rangle \in R$ and all $a \in A$, it holds that

$$s \xrightarrow{a} s' \implies \exists t' \in S \colon t \xrightarrow{a} t' \wedge \langle s', t' \rangle \in R.$$

Two states s and t are called bisimilar if and only if they are related by some bisimulation relation. Notation: $s \sim_l t$.

When replacing the transition relation of an LTS by a 'probabilistic transition relation', the so-called generative probabilistic systems are obtained.

Definition 2.4 A generative probabilistic system is a triple $\langle S, A, \mathrm{P} \rangle$ where S and A are sets and the mapping $\mathrm{P} \colon\ : S \times A \times S \to [0, 1]$ has the property that, for all $s \in S$, it holds that

$$\sum_{a \in A,\ s' \in S} \mathrm{P}(s, a, s') \in \{0, 1\}. \tag{3}$$

Again, we refer to S as the set of states and to A as the set of actions of the system. P is called the probabilistic transition relation. Condition (3) states that, for all $s \in S$, $\mathrm{P}(s, \_, \_)$ is either a probability distribution over $A \times S$ or $\mathrm{P}(s, \_, \_) \equiv 0$, i.e s is a terminating state. As usual, we write $s \xrightarrow{a[p]} s'$ whenever $\mathrm{P}(s, a, s') = p$, and $s \xrightarrow{a} s'$ for $\mathrm{P}(s, a, s') > 0$.

In order to clarify condition (3), let us recall that the sum of an arbitrary family $\{x_i \mid i \in I\}$ of non-negative real numbers is defined as

$$\sum_{i \in I} x_i \ = \ \sup\{ \sum_{i \in J} x_i \mid J \subseteq I, J \text{ finite} \}.$$

Note that, if $\sum_{i \in I} x_i < \infty$, then we also have that the set $\{ x_i \mid i \in I, x_i \neq 0 \}$ is finite or countably infinite.

Definition 2.5 Let $\langle S, A, \mathrm{P} \rangle$ be a generative system. An equivalence relation $R \subseteq S \times S$ is a (strong) bisimulation on $\langle S, A, \mathrm{P} \rangle$ if and only if, for every pair $\langle s, t \rangle \in R$, all $a \in A$ and all equivalence classes $C \in S/R$, it holds that

$$\mathrm{P}(s, a, C) = \mathrm{P}(t, a, C).$$

Here we have put $\mathrm{P}(s,a,C) = \sum_{s' \in C} \mathrm{P}(s,a,s')$. Two states s and t are bisimilar if and only if they are related by some bisimulation relation. Notation: $s \sim_g t$.

Let us now switch to the coalgebraic perspective. It is well-known that the LTSs can be viewed as coalgebras corresponding to the bifunctor

$$\mathcal{L} = \mathcal{P}(\mathcal{I}d \times \mathcal{I}d).$$

Namely, if $\langle S, A, \to \rangle$ is an LTS, then $\langle S, A, \alpha \rangle$, where $\alpha \colon S \to \mathcal{L}_A(S)$ is given by

$$\langle a, s' \rangle \in \alpha(s) \iff s \xrightarrow{a} s',$$

is an $\mathcal{L}_A$ coalgebra, and vice-versa. Also, the generative systems can be considered as coalgebras of the bifunctor

$$\mathcal{G} = \mathcal{D}(\mathcal{I}d \times \mathcal{I}d) + 1.$$

Here, $\mathcal{D}$ denotes the distribution functor on Set, that is,

$$\mathcal{D}X = \{\, \mu \colon X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1 \,\} \text{ and } (\mathcal{D}f)(\mu)(y) = \sum_{f(x)=y} \mu(x)$$

for a set X and a mapping $f \colon X \to Y$ (and $\mu \in \mathcal{D}X, y \in Y$).

If $\langle S, A, P \rangle$ is a generative system, then $\langle S, A, \alpha \rangle$ is a $\mathcal{G}_A$ coalgebra where $\alpha \colon S \to \mathcal{G}_A(S)$ is given by

$$\alpha(s)(a, s') = P(s, a, s'),$$

and vice-versa. Here, the singleton set 1 is interpreted as the set containing the zero-function on $A \times S$. Note that $\alpha(s)$ is the zero-function if and only if s is a terminating state.

The concrete notion of bisimilarity for LTSs and generative systems and the respective coalgebraic definitions coincide. For the case of LTSs a direct proof can be found in [25], for example. For generative systems this fact goes back to [30] where Markov chains were considered, and was treated in [6] for generative systems with finite support.

Here, we describe a general procedure to obtain coincidence results of this kind. It applies to LTSs as well as to generative systems in their full generality. Additionally, we will apply the method for obtaining a concrete characterization, Lemma 2.12, of bisimilarity for a functor discussed in Section 5.

Definition 2.6 Let $R \subseteq S \times T$ be a relation, and $\mathcal{F}$ a Set functor. The relation $\equiv_{\mathcal{F},R} \subseteq \mathcal{F}S \times \mathcal{F}T$, defined by

$$x \equiv_{\mathcal{F},R} y \iff \exists z \in \mathcal{F}R \colon \mathcal{F}\pi_1(z) = x,\ \mathcal{F}\pi_2(z) = y,$$

for $x \in S$, $y \in T$, is called the lifting of R with respect to $\mathcal{F}$.

The following lemma is immediate from Definition 2.2.

Lemma 2.7 *A relation $R \subseteq S \times T$ is a bisimulation for the $\mathcal{F}_A$ systems $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ if and only if $\langle s, t\rangle \in R \implies \alpha(s) \equiv_{\mathcal{F}_A,R} \beta(t)$.* □

Note that the right-hand side of the implication $\langle s, t\rangle \in R \implies \alpha(s) \equiv_{\mathcal{F}_A,R} \beta(t)$ is, in concrete cases, commonly referred to as transfer condition.

In the remainder of this section we gather some results related to the weak preservation of pullbacks. A functor is said to weakly preserve total pullbacks if it transforms any pullback diagram with epi legs into a weak pullback diagram. The restriction to pullbacks with epi legs, rather than arbitrary ones, is a novel technicality, that is needed below.

Lemma 2.8 *If the functor $\mathcal{F}$ weakly preserves total pullbacks and R is an equivalence on S, then the lifting $\equiv_{\mathcal{F},R}$ of R with respect to $\mathcal{F}$ is the pullback in Set of the cospan*

$$\mathcal{F}S \xrightarrow{\mathcal{F}c} \mathcal{F}(S/R) \xleftarrow{\mathcal{F}c} \mathcal{F}S \tag{4}$$

where $c\colon S \to S/R$ is the canonical morphism that maps each element to its equivalence class. □

Suppose that a functor $\mathcal{F}$ weakly preserves total pullbacks and assume that R is an equivalence bisimulation on S, i.e., R is both an equivalence relation and a bisimulation on S, such that $\langle s, t\rangle \in R$. The pullback in Set of the cospan (4) is the set $\{\, \langle x, y\rangle \mid \mathcal{F}c(x) = \mathcal{F}c(y)\,\}$. By Lemma 2.8 this set coincides with the lifted relation $\equiv_{\mathcal{F},R}$. Thus $x \equiv_{\mathcal{F},R} y \iff \mathcal{F}c(x) = \mathcal{F}c(y)$. Therefore, we obtain the transfer condition for the particular notion of bisimulation if we succeed in expressing concretely $(\mathcal{F}c \circ \alpha)(s) = (\mathcal{F}c \circ \alpha)(t)$ in terms of the representation of $\alpha(s)$ and $\alpha(t)$.

For example, consider the LTS functor $\mathcal{L}_A$, which preserves weak pullbacks. For $X \in \mathcal{L}_A(S)$, i.e. $X \subseteq A \times S$, we have $\mathcal{L}_A(c)(X) = \mathcal{P}\langle id_A, c\rangle(X) = \langle id_A, c\rangle(X) = \{\langle a, c(s)\rangle \mid \langle a, s\rangle \in X\}$. Using Lemma 2.7 we get that an equivalence $R \subseteq S \times S$ is a coalgebraic bisimulation for an LTS $\langle S, A, \alpha\rangle$ if and only if

$$\langle s, t\rangle \in R \implies \{\, \langle a, c(s')\rangle \mid \langle a, s'\rangle \in \alpha(s)\,\} = \{\, \langle a, c(t')\rangle \mid \langle a, t'\rangle \in \alpha(t)\,\}$$

or equivalently,

$$\langle s, t \rangle \in R \implies (\; s \xrightarrow{a} s' \implies \exists t' \in S \colon t \xrightarrow{a} t' \wedge \langle s', t' \rangle \in R \;).$$

Thus, we have obtained that an equivalence relation R is an coalgebraic bisimulation with respect to $\mathcal{L}_A$ in the sense of Definition 2.2 if and only if it is a concrete bisimulation in the sense of Definition 2.3.

Often preservation of weak pullbacks is required for the functors to be 'well-behaved', for example, in order that bisimilarity is transivtive. It can easily be seen that the weaker condition of weakly preserving total pullbacks suffices for bisimilarity to be an equivalence already. We need to relax the condition of preservation of weak pullbacks, since we will need a characterization of bisimulation for a functor that weakly preserves total pullbacks, but does not preserve weak pullbacks.

Next, we focus on the weak pullback preservation of the functor $\mathcal{G}_A$. For the functor defining generative systems with finite support weak pullback preservation was proven by De Vink and Rutten [30], using the graph theoretical max-flow min-cut theorem, and by Moss [19], using an elementary matrix fill-in property. We follow the latter approach for arbitrary, infinite, matrices here.

Lemma 2.9 *The functor $\mathcal{D}$ preserves weak pullbacks.* □

The proof of Lemma 2.9 relies on the following 'fill-in' property, the proof of which can be found in [29].

Lemma 2.10 *Let C and D be sets and let $\phi : C \to [0,1]$ and $\psi : D \to [0,1]$ be such that*

$$\sum_{x \in C} \phi(x) = \sum_{y \in D} \psi(y) < \infty. \tag{5}$$

Then there exists a function $\mu : C \times D \to [0,1]$ such that

$$\sum_{y \in D} \mu(x_0, y) = \phi(x_0) \qquad \text{and} \qquad \sum_{x \in C} \mu(x, y_0) = \psi(y_0) \tag{6}$$

for any $x_0 \in C$ and any $y_0 \in D$. □

Using Lemma 2.8, it can be shown that an equivalence relation R on a set S is a coalgebraic bisimulation for the generative system $\langle S, A, \alpha \rangle$ with respect to the functor $\mathcal{G}_A$ if and only if it is a concrete bisimulation according to Definition 2.5.

For our treatment of weak probabilistic bisimulation, we need to consider one more type of systems. Let $\mathcal{G}^*$ be the bifunctor defined by

$$\mathcal{G}^*(A, S) = \mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1]$$

for sets S and A, and,

$$\mathcal{G}^* f(\nu) = \nu \circ \langle f_1^{-1}, f_2^{-1} \rangle.$$

for a morphism $f = \langle f_1, f_2 \rangle \colon A \times S \to B \times T$ (with $\nu \in \mathcal{G}^*(A, S)$). Consider the Set functor $\mathcal{G}^*_A$ corresponding to $\mathcal{G}^*$. Then clearly, $\mathcal{G}^*_A(S) = \mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1]$ and for a mapping $f \colon S \to T$, $\mathcal{G}^*_A f(\nu) = \nu \circ \langle id_A, f^{-1} \rangle$. We seek to characterize equivalence bisimulations for this functor. In order to apply Lemma 2.8 we need the following property.

Lemma 2.11 *The functor $\mathcal{G}^*_A$ weakly preserves total pullbacks.* □

Remarkably, $\mathcal{G}^*_A$ does not preserve weak pullbacks: Choose a set X with $|X| \geq 3$. Fix $x_0 \in X$. Let $Z = \{1, 2, 3\}$ and consider the cospan $X \xrightarrow{f} Z \xleftarrow{g} X$ for the maps $f, g \colon X \to Z$ given by

$$f(x) = \begin{cases} 2 & x = x_0 \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 2 & x = x_0 \\ 3 & \text{otherwise.} \end{cases}$$

The Set pullback of this cospan is $\{\langle x_0, x_0 \rangle\}$, and it is not transformed into a weak pullback by $\mathcal{G}^*_A$.

Let R be an equivalence relation on a set S. A subset $M \subseteq S$ is an R-saturated set if for all $s \in M$ the whole equivalence class of s is contained in M. We use $\mathrm{Sat}(R)$ to denote the set of all R-saturated subsets of S. Actually, M is a saturated set if and only if $M = \cup_{i \in I} C_i$ for a collection $\{C_i\}_{i \in I}$ in S/R. Hence, there is a one-to-one correspondence between R-saturated sets and elements of $\mathcal{P}(S/R)$.

Now, consider the pullback P of the cospan $\mathcal{G}^*_A S \xrightarrow{\mathcal{G}^*_A c} \mathcal{G}^*_A(S/R) \xleftarrow{\mathcal{G}^*_A c} \mathcal{G}^*_A S$. We have

$$\begin{aligned} \langle \mu, \nu \rangle \in P &\iff \mathcal{G}^*_A c(\mu) = \mathcal{G}^*_A c(\nu) \\ &\iff \mu \circ \langle id_A, c^{-1} \rangle = \nu \circ \langle id_A, c^{-1} \rangle \\ &\iff \forall A' \subseteq A, \forall M \subseteq S/R \colon \mu(A', c^{-1}(M)) = \nu(A', c^{-1}(M)) \\ &\iff \forall A' \subseteq A, \forall M \in \mathrm{Sat}(R) \colon \mu(A', M) = \nu(A', M) \end{aligned}$$

since $c^{-1} \colon \mathcal{P}(S/R) \to \mathrm{Sat}(R)$ is a bijection. Hence, we have shown the following characterization.

Lemma 2.12 *An equivalence relation R is a bisimulation for the $\mathcal{G}^*_A$ system $\langle S, A, \alpha\rangle$ if and only if $\langle s, t\rangle \in R \implies \forall A' \subseteq A, \forall M \in \mathrm{Sat}(R)\colon \alpha(s)(A', M) = \alpha(t)(A', M)$.*

3 A coalgebraic definition of weak bisimulation

In this section we present a general definition of weak bisimulation for action type systems. Our definition has been inspired by the definitions of weak bisimulation for concrete systems. Starting point is the idea that a weak bisimulation for a given system arises as strong bisimulation for a system obtained from the original one.

The definition of weak bisimulation consists of two phases. First we define a so-called $*$-*extended system*, that captures the behaviour of the original system for words in A^* instead of actions from A only. The $*$-extension should emerge from the original system in a faithful way. The second phase considers invisibility. Given a subset $\tau \subseteq A$ of invisible actions, we restrict the $*$-extension to visible behaviour only, by defining a so-called *weak-τ-extended system*. In this system, labels are equivalence classes of words. Then a weak bisimulation relation on the original system is a bisimulation relation on the weak-τ-extension.

Definition 3.1 Let $\mathcal{F}$ and $\mathcal{G}$ be two bifunctors. Let Φ be a map assigning to every $\mathcal{F}_A$ coalgebra $\langle S, A, \alpha\rangle$, a $\mathcal{G}_{A^*}$ system $\langle S, A^*, \alpha'\rangle$, with the same set of states, such that the following conditions are met:

(i) Φ is injective, i.e. $\Phi(\langle S, A, \alpha\rangle) = \Phi(\langle S, A, \beta\rangle) \Rightarrow \alpha = \beta$;

(ii) Φ preserves and reflects bisimilarity, i.e. $s \sim t$ in the system $\langle S, A, \alpha\rangle$ if and only if $s \sim t$ in the system $\Phi(\langle S, A, \alpha\rangle)$.

Then Φ is called a $*$-translation from $\mathcal{F}$ to $\mathcal{G}$, notation $\Phi\colon \mathcal{F} \xrightarrow{*} \mathcal{G}$, and we say that $\Phi(\langle S, A, \alpha\rangle)$ is a $*$-extension of $\langle S, A, \alpha\rangle$.

The conditions (i) and (ii) in Definition 3.1 guarantee that the original system is 'embedded' in its $*$-extension, cf. [6,28]. At first sight, it may seem counter-intuitive that the $*$-translation yields a system of another type, viz. of the bifunctor $\mathcal{G}$ rather than of the bifunctor $\mathcal{F}$. However, this extra freedom is crucial in cases where the starting functor is not expressive enough to allow for a $*$-extension (cf. Section 5 on generative systems).

Previous work, [6, Theorem 3.9] provides a way of obtaining $*$-translations. Namely, if $\lambda\colon \mathcal{F}_A \Rightarrow \mathcal{G}_{A^*}$ is a natural transformation with injective components and the functor $\mathcal{F}_A$ preserves weak pullbacks, then the induced functor is a $*$-translation (cf. equation (2)). However, considering $*$-translations emerging from natural transformations only, is not enough. Actually, the $*$-translations in Section 4 and Section 5 are not emerging from natural transformations.

Next, we address how to deal with a subset $\tau \subseteq A$ of invisible actions. The hiding function $h_\tau \colon A^* \to (A \setminus \tau)^*$ is the homomorphism such that $h_\tau(a) = a$ if $a \notin \tau$ and $h_\tau(a) = \varepsilon$ for $a \in \tau$ (with ε denoting the empty word). Consider the set $A_\tau = (A \setminus \tau)^*$. According to Proposition 2.1, the hiding function $h_\tau : A^* \to A_\tau$ determines a natural transformation such that

$$\eta^\tau \colon \mathcal{G}_{A^*} \Rightarrow \mathcal{G}_{A_\tau} \text{ and } \eta^\tau_S = \mathcal{G}\langle h_\tau, id_S\rangle.$$

Let the functor $\Psi_\tau \colon \mathsf{Coalg}_{\mathcal{G}}^{A^*} \to \mathsf{Coalg}_{\mathcal{G}}^{A_\tau}$ be induced by the natural transformation η^τ, i.e. $\Psi_\tau(\langle S, A^*, \alpha'\rangle) = \langle S, A_\tau, \alpha''\rangle$ where $\alpha'' = \eta^\tau_S \circ \alpha'$ and $\Psi_\tau f = f$ (see equation (2)). The weak τ-translation W_τ for a $*$-translation Φ and set of invisible actions τ is then defined as the composition $W_\tau = \Psi_\tau \circ \Phi$.

Definition 3.2 Let $\mathcal{F}$, $\mathcal{G}$ be two bifunctors, $\Phi \colon \mathcal{F} \xrightarrow{*} \mathcal{G}$ a $*$-translation and $\tau \subseteq A$. A relation $R \subseteq S \times T$ is a weak bisimulation for two $\mathcal{F}$-systems $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ with respect to Φ and τ if and only if R is a bisimulation for the coalgebras $W_\tau(\langle S, A, \alpha\rangle)$ and $W_\tau(\langle T, A, \beta\rangle)$. Two states $s \in S$ and $t \in T$ are weakly bisimilar with respect to Φ and τ, notation $s \approx_\tau t$, if they are related by some weak bisimulation with respect to Φ and τ.

The next proposition states that a weak bisimilarity relation $\approx_\tau$ in the sense of Definition 3.2 satisfies the basic properties of a weak bisimilarity relation.

Proposition 3.3 *Let $\mathcal{F}$, $\mathcal{G}$ be two bifunctors, $\Phi \colon \mathcal{F} \xrightarrow{*} \mathcal{G}$, $\langle S, A, \alpha\rangle$ an $\mathcal{F}_A$ coalgebra, $\tau \subseteq A$ and let $\approx_\tau$ denote the weak bisimilarity on $\langle S, A, \alpha\rangle$ with respect to Φ and τ. Then the following hold:*

- (i) $\sim \subseteq \approx_\tau$ *for any $\tau \subseteq A$, i.e. strong bisimilarity implies weak.*
- (ii) $\sim = \approx_\emptyset$, *i.e. strong bisimilarity is weak bisimilarity in absence of invisible actions.*
- (iii) $\tau_1 \subseteq \tau_2 \Rightarrow \approx_{\tau_1} \subseteq \approx_{\tau_2}$ *for any $\tau_1, \tau_2 \subseteq A$, i.e. when more actions are invisible the weak bisimilarity relation gets coarser.* □

It should be noted that in the proof of the above proposition presented in [29] all requirements introduced in Definition 3.1 have been exploited. Therefore, it seems that these requirements are the natural ones. Further justification for Definition 3.2 will be collected in the following two sections where the specific cases for coalgebraic weak bisimulation in labeled transition systems and generative probabilistic transitions systems are explored.

For further reference, we introduce some more notation. For any $w \in A_\tau = (A \setminus \tau)^*$, we denote $B_w = h_\tau^{-1}(\{w\}) \subseteq A^*$. We refer to the sets B_w as blocks. Note that $B_w = \tau^* a_1 \tau^* \cdots \tau^* a_k \tau^*$ for $w = a_1 \ldots a_k \in (A \setminus \tau)^*$.

Traiditionally, the subset of invisible actions is just a singleton consisting of the silent step τ only. However, apart from the mathematical appeal of a general definition that can handle multiple invisible actions, such flexibility is also advantageous, e.g. when dealing with weak bisimulation for Segala-systems (cf. [27,26]).

4 Weak bisimulation for labelled transition systems

In this section we recast the standard definition of weak bisimulation of Milner [17,18]. We provide a $*$-translation for which the coalgebraic formulation of weak bisimulation coincides with the concrete one.

Definition 4.1 Let $\langle S, A, \rightarrow \rangle$ be an LTS. Assume $\tau \in A$ is an invisible action. An equivalence relation $R \subseteq S \times S$ is a weak bisimulation on $\langle S, A, \rightarrow \rangle$ if and only if whenever $\langle s, t \rangle \in R$ then

$$s \xrightarrow{a} s' \implies \exists t' \in S \colon t \xrightarrow{\tau}{}^* \circ \xrightarrow{a} \circ \xrightarrow{\tau}{}^* t' \wedge \langle s', t' \rangle \in R,$$

for all $a \in A \setminus \{\tau\}$, and

$$s \xrightarrow{\tau} s' \implies \exists t' \in S \colon t \xrightarrow{\tau}{}^* t' \wedge \langle s', t' \rangle \in R.$$

Two states s and t are called weakly bisimilar if and only if they are related by some weak bisimulation relation, notation $s \approx_l t$.

Let $\mathcal{L}$, $\mathcal{L}_A$ be the functors for LTSs with set of labels A, as introduced in Section 2. The $*$-translation Φ below captures the natural extension of the transition relations from actions to finite strings of actions.

Definition 4.2 Let Φ assign to any LTS, i.e. an $\mathcal{L}_A$ coalgebra $\langle S, A, \alpha \rangle$, the $\mathcal{L}_{A^*}$ coalgebra $\langle S, A^*, \alpha' \rangle$ where, for $w = a_1 \ldots a_k \in A^*$, $\langle w, s' \rangle \in \alpha'(s)$ if and only if there exist states $s_1, \ldots, s_{k-1} \in S$ such that $s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{k-1} \xrightarrow{a_k} s'$.

We have the following correspondence result.

Theorem 4.3

(i) *The assignment Φ given by Definition 4.2 is a $*$-translation.*

(ii) *Let $\langle S, A, \alpha\rangle$ be an LTS. Let $\tau \in A$ be an invisible action and $s, t \in S$ any two states. Then $s \approx_{\{\tau\}} t$ according to Definition 3.2 with respect to Φ and $\{\tau\}$ if and only if $s \approx_l t$ according to Definition 4.1.* $\square$

The proof of Theorem 4.3 is straightforward and can be found in [29].

5 Weak bisimulation for generative systems

In this section we deal with generative probabilistic transition systems and their weak bisimilarity. Inspired by the work of Baier and Hermanns [3,2,4], we provide a $*$-translation on which we base a notion of weak bisimulation for generative probabilistic systems. We show that our definition coincides with the definition of Baier and Hermanns. Unlike in the case of LTSs, here the generality of our set-up is necessary as the $*$-translation really yields images outside of the class of generative systems.

Let $\langle S, A, \mathrm{P}\rangle$, be a generative system. A finite path π of $\langle S, A, \mathrm{P}\rangle$ is an alternating sequence $\pi \equiv s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{k-1} \xrightarrow{a_k} s_k$ such that $k \in \mathbb{N}_0$, $s_i \in S$, $a_j \in A$ and $\mathrm{P}(s_i, a_{i+1}, s_{i+1}) > 0$. We put $\mathrm{length}(\pi) = k$, $\mathrm{first}(\pi) = s_0$, $\mathrm{last}(\pi) = s_k$, $\mathrm{trace}(\pi) = a_1 a_2 \cdots a_k$. We use ε to denote the empty path starting in s. An infinite path π of $\langle S, A, \mathrm{P}\rangle$ is an alternating sequence $\pi \equiv s \xrightarrow{a_1} slong_1 \xrightarrow{a_2} s_2 \cdots$ where for all $i \geq 1$, $\mathrm{P}(s_i, a_{i+1}, s_{i+1}) > 0$. The first element of an non-empty path is indicated by $\mathrm{first}(\pi)$. A complete path is either an infinite path or a finite path ending in a terminating state.

The sets of all (finite or infinite) paths, of all finite paths and of all complete paths will be denoted by Paths, FPaths and CPaths, respectively. Moreover, for $s \in S$, we write

$$\mathrm{Paths}(s) = \big\{\pi \in \mathrm{Paths} \mid \mathrm{first}(\pi) = s\big\}$$

and similarly, we use $\mathrm{FPaths}(s)$ and $\mathrm{CPaths}(s)$. The set $\mathrm{Paths}(s)$ is partially ordered by the prefix relation $\preccurlyeq$. Note that $\varepsilon \preccurlyeq \pi$ for all $\pi \in \mathrm{Paths}(s)$. We stress that for any state $s \in S$, the set $\mathrm{FPaths}(s)$ is at most countable. For a finite path $\pi \in \mathrm{FPaths}(s)$, we put

$$\pi\!\uparrow = \{\, \xi \in \mathrm{CPaths}(s) \mid \pi \preccurlyeq \xi \,\}.$$

We call $\pi\!\uparrow$ the cone of complete paths for π. Let

$$\Gamma = \{\, \pi\!\uparrow \mid \pi \in \mathrm{FPaths}(s) \,\}$$

denote the set of all cones starting in s. Note that any two cones $\pi_1\!\uparrow$ and $\pi_2\!\uparrow$ are either disjoint or one is a subset of the other or vice-versa. From this and the fact that FPaths(s) is at most countable we see that $\Gamma \cup \{\emptyset\}$ has the properties:

– it contains the empty set,

– it is closed with respect to intersection,

– for any two elements $X, Y \in \Gamma \cup \{\emptyset\}$ the difference $X \setminus Y$ can be written as a countable union of elements of $\Gamma \cup \{\emptyset\}$.

We define a function Prob: $\Gamma \cup \{\emptyset\} \to \mathbb{R}$ by putting $\mathrm{Prob}(\emptyset) = 0$, $\mathrm{Prob}(\varepsilon\uparrow) = 1$ and

$$\mathrm{Prob}(\pi\uparrow) = \mathrm{P}(s, a_1, s_1) \cdot \mathrm{P}(s_1, a_2, s_2) \cdot \cdots \cdot \mathrm{P}(s_{k-1}, a_k, s_k),$$

for $\pi = s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{k_1} \xrightarrow{a_k} s_k$. Let us remark that the function Prob is indeed well-defined. This function has the following properties:

– $\mathrm{Prob}(\emptyset) = 0$,

– $\mathrm{Prob}(X) \leq \mathrm{Prob}(Y)$ whenever $X \subseteq Y$ for $X, Y \in \Gamma \cup \{\emptyset\}$,

– If $X \in \Gamma \cup \{\emptyset\}$ can be written as a at most countable disjoin union $X = \cup_n X_n$ of elements $X_n \in \Gamma \cup \{\emptyset\}$ then $\mathrm{Prob}(X) = \sum_n \mathrm{Prob}(X_n)$.

It follows from [31] [5] that Prob can be uniquely extended to a measure on the σ-algebra generated by $\Gamma \cup \{\emptyset\}$. Since, by definition, $\mathrm{Prob}(\varepsilon\uparrow) = 1$, this is a probability measure.
If $\Pi \subseteq$ FPaths(s), we denote by $\Pi\uparrow \subseteq$ CPaths(s) the set

$$\Pi\uparrow = \bigcup_{\pi \in \Pi} \pi\uparrow.$$

Note that $\Pi\uparrow$ belongs to the σ-algebra generated by $\Gamma \cup \{\emptyset\}$.

The induced measure yields a function Prob: $\mathcal{P}(\mathrm{FPaths}(s)) \to \mathbb{R}$ by defining $\mathrm{Prob}(\Pi)$ as the measure of the collection $\Pi\uparrow$. This function is, in general, not additive; we only have $\mathrm{Prob}(\Pi) \leq \sum_{\pi\in\Pi} \mathrm{Prob}(\{\pi\})$. Nevertheless, for sets Π which are minimal in a certain sense, equality still holds. Here, we call a set $\Pi \subseteq$ FPaths(s) minimal, notation min(Π), if and only if for any two different $\pi_1, \pi_2 \in \Pi$ we neither have $\pi_1 \preccurlyeq \pi_2$ nor $\pi_2 \preccurlyeq \pi_1$. If min(Π), then $\mathrm{Prob}(\Pi) = \sum_{\pi\in\Pi} \mathrm{Prob}(\pi\uparrow)$. Every set $\Pi\uparrow$ can also be generated by a minimal set. For $\Pi \subseteq$ FPaths(s), let

$$\Pi\downarrow = \{\pi \in \Pi \mid \forall \pi' \in \Pi \colon \pi' \not\prec \pi\}$$

[5] The point of referring to [31] rather than to a more popular texts such as Halmos [13], is that the version of the extension theorem in [31] applies to semi-rings where set differences can be represented as countable unions rather than finite unions of elements of the semi-ring.

then $\Pi\!\uparrow = (\Pi\downarrow)\!\uparrow$. Finally, let $s \in S, S' \subseteq S$ and $W \subseteq A^*$. We put

$$s \xrightarrow{W} S' = \{\pi \in \mathrm{FPaths}(s) \mid \mathrm{first}(\pi) = s, \mathrm{last}(\pi) \in S' \subseteq S, \mathrm{trace}(\pi) \in W \subseteq A^*\}\downarrow,$$

and write $\mathrm{Prob}(s, W, S') = \mathrm{Prob}(s \xrightarrow{W} S')$.

We proceed by presenting the $*$-translation for generative probabilistic systems as caputered by the bifunctor $\mathcal{G}^*$.

Definition 5.1 Let Φ^g assign to every generative system i.e., to any $\mathcal{G}_A$ coalgebra $\langle S, A, \alpha\rangle$, the $\mathcal{G}^*_{A^*}$ coalgebra $\langle S, A^*, \alpha'\rangle$ where for $W \subseteq A^*$ and $S' \subseteq S$, $\alpha'(s)(W, S') = \mathrm{Prob}(s, W, S')$.

Theorem 5.2 *The assignment Φ^g is a $*$-translation.* □

The proof is provided in [29]. The main difficulty is the preservation of bisimulation. For this a detailed analysis is conducted of the collection of paths of the form $s \xrightarrow{a_1} C_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} C_k$ with $C_1, \ldots, C_k$ equivalence classes of states modulo a bisimulation relation.

In the context of Φ^g, the weak-τ-system is of the form

$$\Psi_\tau \circ \Phi^g(\langle S, A, \alpha\rangle) = \Psi_\tau(\langle S, A^*, \alpha'\rangle) = \langle S, A_\tau, \alpha''\rangle$$

where $\alpha''(s)\colon \mathcal{P}(A_\tau) \times \mathcal{P}(S) \to [0,1]$ is given by

$$\alpha''(s) = \ \eta^\tau_S(\alpha'(s)) = \ \mathcal{G}^*\langle h_\tau, id_S\rangle(\alpha'(s)) = \ \alpha'(s) \circ \langle h_\tau^{-1}, id_S\rangle.$$

Hence for $X \subseteq A_\tau = (A \setminus \tau)^*$ and $S' \subseteq S$, we have that

$$\alpha''(s)(X, S') = \alpha'(s)(h_\tau^{-1}(X), S') = \alpha'(s)(\bigcup_{w\in X} B_w, S') = \mathrm{Prob}(s, \bigcup_{w\in X} B_w, S').$$

Therefore, from Lemma 2.12 we get that an equivalence relation R is a weak-τ-bisimulation with respect to Φ^g and τ on the generative system $\langle S, A, \alpha\rangle$ if and only if $\langle s, t\rangle \in R$ implies that, for any collection $\{B_i\}_{i\in I}$ of blocks and any collection $\{C_j\}_{j\in J}$ of classes, it holds that

$$\mathrm{Prob}(s, \bigcup_{i\in I} B_i, \bigcup_{j\in J} C_j) = \mathrm{Prob}(t, \bigcup_{i\in I} B_i, \bigcup_{j\in J} C_j). \tag{7}$$

Note that $\cup_{i\in I} B_i$ is a $\ker(h_\tau)$-saturated set and that $\cup_{j\in J} C_j$ is an R-saturated set.

Next we recall the original definition of weak bisimulation for generative systems by Baier and Hermanns [3,2,4].

Definition 5.3 Let $\langle S, A, P\rangle$ be a generative probabilistic system. Let $\tau \in A$ be an invisible action. An equivalence relation $R \subseteq S \times S$ is a weak bisimulation on $\langle S, A, P\rangle$ if and only if, for every pair $\langle s, t\rangle \in R$, all actions $a \in A$ and for all equivalence classes $C \in S/R$, it holds that

$$\mathrm{Prob}(s, \tau^* \hat{a} \tau^*, C) = \mathrm{Prob}(t, \tau^* \hat{a} \tau^*, C), \tag{8}$$

where $\hat{a} = a$ for $a \in A \setminus \{\tau\}$ and $\hat{\tau} = \varepsilon$, the empty word. Two states s and t are weakly bisimilar if and only if they are related by some weak bisimulation relation. Notation $s \approx_g t$.

We have the following correspondence result.

Theorem 5.4 *Let $\langle S, A, \alpha\rangle$ be a generative system. Let $\tau \in A$ be an invisible action and $s, t \in S$ any two states. Then $s \approx_{\{\tau\}} t$ according to Definition 3.2 with respect to Φ^g and $\{\tau\}$ if and only if $s \approx_g t$ according to Definition 5.3.*

The sufficiency part of the theorem holds trivially, having Definition 5.3 and Equation (7) in mind, since τ^* as well as $\tau^* a \tau^*$, for any $a \in A \setminus \{\tau\}$, is a $\ker(h_{\{\tau\}})$-saturated set. Additionally, each R-equivalence class is an R-saturated set. Hence $\approx_{\{\tau\}}$ is at least as strong as $\approx_g$ is. The necessity proof is more involved. In [29] a series of lemmas shows that (8) implies (7). The difficulty is that the expression $\mathrm{Prob}(s, W, M)$ is not additive in its second nor in its third argument. The proofs exploit combinatorial arguments requiring a detailed analysis of the geometry of paths. Note that this technical obstacle does not occur in the qualitative setting of LTSs.

6 Concluding remarks

In this paper we have proposed a coalgebraic definition of weak bisimulation for action-type systems. For its justification we have considered the cases of the familiar labelled transition systems and of generative probabilistic systems and have argued that the coalgebraic notion coincides with the concrete definitions. Additionally, the paper also comprises a few other, smaller contributions.

This paper builds on earlier work jointly with Falk Bartels [6,7]. In Section 2 we have discussed a general method for obtaining correspondence results for coalgebraic versus concrete bisimulations. Our presentation generalizes the direct approach with explicit proofs in the work mentioned. The main idea is to tie up the reformulation of coalgebraic bisimulation in terms of the lifted bisimulation relation $\equiv_{\mathcal{F},R}$ and the pullback of a particular cospan (cf. Lemma 2.8). The method works for any functor that weakly preserves total pullbacks, i.e. pullbacks with epi legs, a condition weaker than weak pullback preservation.

Our handling of probabilistic distributions avoids restricting the cardinality of the support set, a fact of some technical interest. The results hold for arbitrary discrete distributions captured by the functor $\mathcal{D}$ of Section 2. Although we do not impose cardinality restrictions on the state spaces considered, generative probabilistic system are discrete in nature. The work of Baier and Hermanns treats finite systems only, also because of the algorithmic considerations addressed in [3,4]. As we do not touch upon such matter here, the definitions, both concrete and coalgebraic, are given for systems of arbitrary size.

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